Definition Let F be a field. A non-empty set V is a vector space over the field F if the following axioms hold:

(1) An operation called *addition* is defined on the set V: to any pair of $\mathbf{v}, \mathbf{w} \in V$ we assign a unique element in V denoted by $\mathbf{v} + \mathbf{w}$.

(2) Addition is associative: any three elemetns $\mathbf{v}, \mathbf{w}, \mathbf{z} \in V$ satisfy
$$(\mathbf{v} + \mathbf{w}) + \mathbf{z} = \mathbf{v} + (\mathbf{w} + \mathbf{z}).$$

(3) Addition is commutative: any two elements $\mathbf{v}, \mathbf{w} \in V$ satisfy
$$\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}.$$

(4) There exists a *zero* element $\mathbf{0} \in V$: every $\mathbf{v} \in V$ satisfies
$$\mathbf{0} + \mathbf{v} = \mathbf{v} + \mathbf{0} = \mathbf{v}.$$

(5) Every element has a negative (additive inverse): to every $\mathbf{v} \in V$ there exists an element $-\mathbf{v} \in V$ satisfying
$$\mathbf{v} + (-\mathbf{v}) = (-\mathbf{v}) + \mathbf{v} = \mathbf{0}.$$

(6) An operation called *multiplication by scalars* is defined between the field F and the set V: to every $\lambda \in F$ and $\mathbf{v} \in V$ we assign a unique element in V denoted by $\lambda\mathbf{v}$.

(7) All $\lambda, \mu \in F$ and $\mathbf{v} \in V$ satisfy
$$(\lambda + \mu)\mathbf{v} = \lambda\mathbf{v} + \mu\mathbf{v}.$$

(8) All $\lambda \in F$ and $\mathbf{v}, \mathbf{w} \in V$ satisfy
$$\lambda(\mathbf{v} + \mathbf{w}) = \lambda\mathbf{v} + \lambda\mathbf{w}.$$

(9) All $\lambda, \mu \in F$ and $\mathbf{v} \in V$ satisfy
$$(\lambda\mu)\mathbf{v} = \lambda(\mu\mathbf{v}).$$

(10) Every $\mathbf{v} \in V$ satisfies
$$1\mathbf{v} = \mathbf{v},$$
where 1 is the multiplicative identity i the field F (i.e., $1\lambda = \lambda 1 = \lambda$ for every $\lambda \in F$).

Note: The elements of V are called vectors so the vector can mean polynomials, matrices, functions, 2-dimensional vectors, 3-dimensional vectors, etc. depending on the set V. The elements of the field F are called scalars. When we talk about a vector space V, we must always specify which field F where scalars come from. In other words, A vector space always comes as a pair: a nonempty set together with a field F. When $F = \mathbb{R}$, we call V a *real* vector space. When $F = \mathbb{C}$, we call V a *complex* vector space.

Examples of Vector Spaces

(1) The plane or space vectors starting from the origin form a real vector space under the ordinary vector addition and multiplication of a vector by a real numbers.

(2) F^k is a vector space over the field F if the operations are performed componentwise. The previous example is the special case $F = \mathbb{R}$ and $k = 2$ or $k = 3$. When $F = \mathbb{R}$ and $k = 1$, we have $\mathbb{R}$ which is a vector space over itself.

(3) The set of $k \times n$ matrices with entries in F is a vector space over F under the usual addition and multiplication by scalars of matrices. This set is denoted by $F^{k\times n}$ or $M_{k\times n}(F)$. The previous example is the special case $n = 1$.

(4) For a fixed n, the set of polynomials with real coefficients of degree $\leq n$ is a real vector space under the usual operations.

(5) Remember infinite sequences you came across in Calculus 2? The set of infinite sequences of real numbers forms a vector space over $\mathbb{R}$ with the usual operations.

(6) The set of functions from $\mathbb{R}$ to $\mathbb{R}$ forms a vector space under the usual operations $f + g : \alpha \mapsto f(\alpha) + g(\alpha)$ and $\lambda f : \alpha \mapsto \lambda f(\alpha)$.

Consequences of the Axioms

(1) The zero element (vector) is unique.

(2) The negative of a vector is unique.

(3) We can perform subtraction, i.e., to every pair of vectors $\mathbf{v}, \mathbf{w} \in V$, there exists a unique vector $\mathbf{z} \in V$ satisfying $\mathbf{w} + \mathbf{z} = \mathbf{v}$, namely $\mathbf{z} = \mathbf{v} + (-\mathbf{w})$. We denote $\mathbf{z}$ by $\mathbf{v} - \mathbf{w}$.

(4) When adding more than two vectors, we can omit the parentheses and order the vectors arbitrarily as addition is associative and commutative.

Theorem

(1) $\lambda\mathbf{0} = \mathbf{0}$ for all $\lambda \in F$.

(2) $0\mathbf{v} = \mathbf{0}$ for every $\mathbf{v} \in V$ where 0 is the zero element in the field F.

(3) $(-1)\mathbf{v} = -\mathbf{v}$ for every $\mathbf{v} \in V$ where -1 is the negative of the multiplicative identity in the field F.

(4) If $\lambda\mathbf{v} = \mathbf{0}$, then $\lambda = 0$ or $\mathbf{v} = \mathbf{0}$.

Note: We will always assume that the coefficients of polynomials and the terms of sequences are real numbers unless otherwise specified.

Question: Which of the following sets are real vector spaces?

(1) The set of polynomials divisible by $x^3 + 1$.

(2) The set of polynomials that leave a constant remainder when divided by $x^3 + 1$.

(3) The set of polynomials whose sum of coefficients is 0.

(4) The set of polynomials whose every coefficient is a rational number.

(5) The set of polynomials which have a real root.

(6) For a fixed n, the set of polynomials of degree n.

(7) The set of bounded sequences.

(8) The set of convergent sequences.

(9) The set of non-decreasing sequences.

(10) The set of monotone sequences

(11) The set of geometric sequences (including the 0 sequence)

(12) The set of continuous functions from $\mathbb{R}$ to $\mathbb{R}$.

(13) The set of even functions from $\mathbb{R}$ to $\mathbb{R}$.

(14) The set of functions from $\mathbb{R}$ to $\mathbb{R}$ such that $f(\pi)$ is an integer.